

# CHOQUET ORDER AND HYPERRIGIDITY FOR FUNCTION SYSTEMS

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**ABSTRACT.** We establish a new characterization of the Choquet order on the space of probability measures on a compact convex set. The characterization is dilation-theoretic, meaning that it relates to the representation theory of positive linear maps on the  $C^*$ -algebra of continuous functions on the set. This yields an extension of Cartier's theorem on dilation of measures that is valid in the non-metrizable setting. As an application, we prove Arveson's hyperrigidity conjecture for function systems, and obtain new approximation theorems for positive maps from commutative  $C^*$ -algebras into  $\mathcal{B}(H)$ .

## 1. INTRODUCTION

In this paper, we establish a new characterization of the Choquet order on the space of probability measures on a compact convex set. The characterization is dilation-theoretic, by which we mean that it relates to the representation theory of positive linear maps on the  $C^*$ -algebra of continuous functions on the set. We develop this new connection between Choquet theory and the theory of operator algebras, and utilize it to establish Arveson's hyperrigidity conjecture for function systems.

Let  $K$  be a compact convex subset of a locally convex vector space. Let  $A(K)$  denote the space of continuous affine functions on  $K$ , and let  $M^+(K)$  denote the space of positive regular Borel measures on  $K$ . Every positive linear functional on  $A(K)$  can be realized as integration against a measure in  $M^+(K)$ , and measures with this property are said to be representing measures for the functional. The Choquet-Bishop-de Leeuw theorem ensures the existence of a representing measure that is supported on the set  $\partial K$  of extreme points of  $K$  when  $K$  is metrizable, and pseudo-supported on  $\partial K$  when  $K$  is non-metrizable.

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The *Choquet order* “ $\prec_c$ ” is a partial order on  $M^+(K)$ . For measures  $\mu, \nu \in M^+(K)$ , we say that  $\mu \prec_c \nu$  if  $\int_K f d\mu \leq \int_K f d\nu$  for every continuous convex function  $f \in C(K)$ . Heuristically, the Choquet order detects when the support of a measure lies closer to the extreme boundary  $\partial K$ . For this reason, a measure that is maximal in the Choquet order is said to be a *boundary measure*.

We introduce a new order on  $M^+(K)$  that we call the *dilation order*, because it is defined in terms of dilation-theoretic notions from the theory of completely positive maps on potentially noncommutative  $C^*$ -algebras. Recently, related ideas played a key role in the solution of Arveson’s conjecture about the existence of the noncommutative Choquet boundary of an operator system [6, 11].

Let  $C(K)$  denote the  $C^*$ -algebra of continuous functions on  $K$ . Every measure  $\mu \in M^+(K)$  has representations  $(\pi, H, \xi)$  consisting of a Hilbert space  $H$ , a distinguished vector  $\xi \in H$  and a  $*$ -representation  $\pi : C(K) \rightarrow \mathcal{B}(H)$  into the  $C^*$ -algebra  $\mathcal{B}(H)$  of bounded linear operators on  $H$  satisfying

$$\int_K f d\mu = \langle \pi(f)\xi, \xi \rangle \quad \text{for all } f \in C(K).$$

For example, a representation of  $\mu$  can always be obtained from the Gelfand-Naimark-Segal (GNS) construction. However, in general  $\mu$  may have many such representations, because we do not insist that  $\xi$  be a cyclic vector.

We write “ $\prec_d$ ” for the *dilation order* on  $M^+(K)$ . For measures  $\mu, \nu \in M^+(K)$ , we say that  $\mu \prec_d \nu$  if some representation  $(\pi, H, \xi)$  of  $\mu$  is dilated by a representation  $(\sigma, L, \nu)$  of  $\nu$ , meaning in particular that  $\pi|_{A(K)}$  can be obtained by compressing  $\sigma$  to a subspace of  $L$ . Our main result is that the dilation order is actually equivalent to the Choquet order.

**Theorem 1.1.** *Let  $K$  be a compact convex subset of a locally convex vector space. The dilation order “ $\prec_d$ ” on  $M^+(K)$  coincides with the Choquet order “ $\prec_c$ ”. Hence a measure in  $M^+(K)$  is a boundary measure if and only if it is maximal in the dilation order.*

A recurring theme in Choquet theory is that working in the general possibly non-metrizable setting is markedly more difficult than working in the metrizable setting. However, by using operator-algebraic methods, we will frequently be able to overcome this difficulty.

For example, in the metrizable setting, a theorem of Cartier relates Choquet order to the notion of dilation of measures. Specifically, Cartier showed that if  $K$  is metrizable and  $\mu, \nu \in M^+(K)$  satisfy

$\mu \prec_c \nu$ , then  $\nu$  is a “dilation” of  $\mu$ , meaning that  $\nu$  can be represented as an integral with respect to  $\mu$  over a family of representing measures. Our methods yield an extension of Cartier’s theorem to the general setting, which seems to have been an open problem for some time (see e.g. [13]).

**Theorem 1.2.** *Let  $K$  be a compact convex subset of a locally convex vector space. Let  $\mu, \nu \in M^+(K)$  satisfy  $\mu \prec_c \nu$ . Then there is a family  $\{\lambda_x\}_{x \in K} \subset M^+(K)$  of probability measures such that*

- (1)  $\lambda_x(a) = a(x)$   $\mu$ -a.e. for all  $a \in A(K)$ ,
- (2)  $f \rightarrow \lambda_x(f)$  is  $\mu$ -measurable for all  $f \in C(K)$ , and
- (3)  $\int f d\nu = \int \lambda_x(f) d\mu$  for all  $f \in C(K)$ .

For a measure  $\mu \in M^+(K)$ , the GNS construction gives rise to a  $*$ -representation  $\pi_\mu : C(K) \rightarrow \mathcal{B}(L^2(\mu))$  of  $C(K)$  by multiplication operators on  $L^2(\mu)$ . We apply Theorem 1.1 to show that boundary measures can be detected from representation-theoretic information.

**Theorem 1.3.** *Let  $K$  be a compact convex subset of a locally convex vector space. Let  $\mu \in M^+(K)$  be a measure with corresponding GNS representation  $\pi_\mu : C(K) \rightarrow \mathcal{B}(L^2(\mu))$ , and let  $\pi_\mu|_{A(K)}$  denote the restriction of  $\pi_\mu$  to the space  $A(K)$  of continuous affine functions on  $K$ . Then  $\mu$  is a boundary measure if and only if  $\pi_\mu$  is the unique extension of  $\pi_\mu|_{A(K)}$  to a completely positive map from  $C(K)$  to  $\mathcal{B}(L^2(\mu))$ .*

Hyperrigidity is a notion of approximation that underlies many important operator-algebraic phenomena. This was first recognized by Arveson, who undertook a comprehensive study of this concept in [7], and made connections with the theory of the noncommutative Choquet boundary.

The most general form of Arveson’s hyperrigidity conjecture concerns operator systems, which are unital self-adjoint subspaces of generally noncommutative  $C^*$ -algebras. An operator system  $A$  that generates a  $C^*$ -algebra  $C$  is said to be *hyperrigid* if whenever  $\pi : C \rightarrow \mathcal{B}(H)$  is a nondegenerate  $*$ -representation of  $C$  on a Hilbert space  $H$  and  $\phi_n : C \rightarrow \mathcal{B}(H)$  is a sequence of unital completely positive maps with the property that  $\lim_n \|\phi_n(a) - \pi(a)\| = 0$  for all  $a \in A$ , then  $\lim_n \|\phi_n(c) - \pi(c)\| = 0$  for all  $c \in C$ .

**Conjecture 1.4** (Arveson’s hyperrigidity conjecture). *An operator system  $A$  that generates a  $C^*$ -algebra  $C$  is hyperrigid if and only if the noncommutative Choquet boundary  $\partial_A C$  of  $A$  coincides with the set of all irreducible representations of  $C$ .*

An operator system  $A$  that generates a commutative  $C^*$ -algebra  $C(X)$  of continuous functions on a compact Hausdorff space  $X$  is said to be a *function system*. In this case, the noncommutative Choquet boundary  $\partial_A C(X)$  of  $A$  coincides with the classical Choquet boundary  $\partial_A X$  of  $A$ , consisting of the points in  $X$  with corresponding point evaluations that restrict to extreme states on  $A$ .

A result of Kadison [15] shows that every function system is order isomorphic to the space  $A(K)$  of continuous affine functions on a compact convex subset  $K$  of a locally compact vector space. In this setting, the Choquet boundary  $\partial_{A(K)} K$  of  $A(K)$  is precisely the set  $\partial K$  of extreme points of  $K$ . This correspondence allows us to apply our Choquet-theoretic results to prove Conjecture 1.4 for function systems.

**Theorem 1.5.** *A function system  $A$  that generates a commutative  $C^*$ -algebra  $C(X)$  is hyperrigid if and only if the Choquet boundary  $\partial_A X$  of  $A$  is all of  $X$ .*

One of Arveson's motivations for studying hyperrigidity is a classical approximation theorem of Korovkin which states that if  $\phi_n : C[0, 1] \rightarrow C[0, 1]$  is a sequence of positive maps satisfying  $\lim_n \|\phi_n(g) - g\| = 0$  for each  $g \in \{1, x, x^2\}$ , then  $\lim_n \|\phi_n(f) - f\| = 0$  for all  $f \in C[0, 1]$ .

Šaškin [20] proved a much more general version of Korovkin's theorem in the setting of a commutative  $C^*$ -algebra  $C(X)$  of continuous functions on a compact metric space  $X$ . A subset  $G \subset C(X)$  is said to be a *Korovkin set* if whenever  $\phi_n : C(X) \rightarrow C(X)$  is a sequence of positive linear maps satisfying  $\lim_n \|\phi_n(g) - g\| = 0$  for all  $g \in G$ , then  $\lim_n \|\phi_n(f) - f\| = 0$  for all  $f \in C(X)$ . Šaškin proved that if  $X$  is metrizable, then a subset  $G \subset C(X)$  that separates points and contains 1 is a Korovkin set if and only if  $\partial_A X = X$ , where  $A = \overline{\text{span}(G \cup G^*)}$  denotes the function system generated by  $G$ .

As an application of Theorem 1.5, we prove a significantly stronger version of Šaškin's theorem. A subset  $G \subset C(X)$  is said to be a *strong Korovkin set* if whenever  $\pi : C(X) \rightarrow \mathcal{B}(H)$  is a  $*$ -representation and  $\phi_n : C(X) \rightarrow \mathcal{B}(H)$  is a sequence of positive maps satisfying  $\lim_n \|\phi_n(g) - \pi(g)\| = 0$  for all  $g \in G$ , then  $\lim_n \|\phi_n(f) - \pi(f)\| = 0$  for all  $f \in C(X)$ .

**Theorem 1.6.** *Let  $C(X)$  denote the  $C^*$ -algebra of continuous functions on a compact Hausdorff space  $X$ . Let  $G \subset C(X)$  be a subset that separates points and contains 1. Then the following are equivalent:*

- (1)  $G$  is a strong Korovkin set.
- (2)  $G$  is a Korovkin set.
- (3)  $\partial_A X = X$ .

Here  $A = \overline{\text{span}(G \cup G^*)}$  denotes the function system generated by  $G$ .

The definition of a strong Korovkin set allows maps with ranges that are possibly noncommutative, and this fact seems to necessitate the use of non-classical methods to prove Theorem 1.6. We explore this issue in some detail. Furthermore, the result is valid without any metrizability assumption.

The ideas developed in this paper have natural noncommutative analogues which we will develop in forthcoming work.

In addition to this introduction, this paper has six sections. In Section 2 we briefly review the requisite background material. In Section 3 we introduce the dilation order and prove that it is equivalent to the Choquet order. In Section 4, we consider some consequences in Choquet theory, including the extension of Cartier's theorem. In Section 5, we consider extensions of positive maps on function systems of continuous affine functions. In Section 6, we consider extensions of positive maps on general function systems and prove Arveson's hyperrigidity conjecture in that setting. In Section 7, we consider applications to approximation theory.

## 2. PRELIMINARIES

Since this work makes considerable use of both Choquet theory and the theory of operator algebras, we will be somewhat generous in providing background material in both areas for the convenience of our readers.

**2.1. Commutative C\*-algebras.** Let  $X$  be a compact Hausdorff space and let  $C(X)$  denote the C\*-algebra of continuous functions on  $X$ . A linear functional  $\alpha : C(X) \rightarrow \mathbb{C}$  is said to be unital if  $\alpha(1) = 1$  and positive if  $\alpha(f) \geq 0$  for every non-negative function  $f \in C(X)$ . If  $\alpha$  is both unital and positive, then it is said to be a *state*.

Let  $M^+(X)$  denote the space of positive regular Borel measures on  $X$ , and let  $P(X)$  denote the space of regular Borel probability measures on  $X$ . By the Riesz-Markov-Kakutani representation theorem, positive linear functionals on  $C(X)$  correspond to measures in  $M^+(X)$ , and in particular, states on  $C(X)$  correspond to probability measures in  $P(X)$ .

For  $\mu \in M^+(X)$ , the corresponding positive linear functional on  $C(X)$  is defined by

$$\mu(f) = \int_X f d\mu \quad \text{for } f \in C(X).$$

**Definition 2.1.** A *representation of  $\mu$*  is a tuple  $(\pi, H, \xi)$  consisting of a  $*$ -representation  $\pi : C(X) \rightarrow \mathcal{B}(H)$  on a Hilbert space  $H$  and a distinguished vector  $\xi \in H$  such that

$$\mu(f) = \langle \pi(f)\xi, \xi \rangle \quad \text{for all } f \in C(X).$$

We will write  $(\pi_\mu, L^2(\mu), 1_\mu)$  for the representation obtained from the Gelfand-Naimark-Segal (GNS) construction for the positive functional  $\mu$ , where  $L^2(\mu) = L^2(X, \mu)$ ,  $1_\mu$  denotes the constant function 1 considered as an element of  $L^2(\mu)$ , and  $\pi_\mu : C(X) \rightarrow \mathcal{B}(L^2(\mu))$  is defined by

$$\pi_\mu(f)h = fh \quad \text{for } f \in C(X), h \in L^2(\mu).$$

The GNS representation of  $\mu$  is minimal, in the sense that if  $(\pi, H, \xi)$  is another representation of  $\mu$ , then the restriction of  $\pi$  to the cyclic invariant subspace for  $\pi$  generated by  $\xi$  is unitarily equivalent to  $\pi_\mu$  via a unitary that maps  $\xi$  to  $1_\mu$ .

It is a standard fact from the theory of representations of  $C^*$ -algebras that every  $*$ -representation  $\pi : C(X) \rightarrow \mathcal{B}(H)$  can be written as a direct sum of cyclic  $*$ -representations. Furthermore, every cyclic  $*$ -representation is unitarily equivalent to the GNS representation  $\pi_\mu$  for some measure  $\mu \in M^+(X)$ .

For a compact subset  $C \subset X$ , we will say that a  $*$ -representation  $\pi$  of  $C(X)$  is *supported on  $C$*  if there are measures  $(\mu_i)_{i \in I}$  in  $M^+(X)$  such that  $\pi$  is unitarily equivalent to the direct sum  $\bigoplus_{i \in I} \pi_{\mu_i}$ , and each  $\mu_i$  is supported on  $C$ .

**2.2. Function systems.** The notion of a function system was introduced by Kadison in [15]. An *abstract function system*  $A$  is an ordered normed vector space that is positively generated, i.e.  $A = A^+ - A^+$ , and has a distinguished archimedean order unit  $1_A$  such that the norm on  $A$  is determined via the formula

$$\|a\| = \inf\{\lambda > 0 : -\lambda 1_A \leq a \leq \lambda 1_A\} \quad \text{for } a \in A.$$

We will consider function systems over the complex numbers. Although the literature often considers function systems over the real numbers, results in the complex case are readily derived from the real case.

A linear functional  $\alpha : A \rightarrow \mathbb{C}$  is said to be *unital* if  $\alpha(1_A) = 1$  and *positive* if  $\alpha(A^+) \subset \mathbb{R}^+$ . If  $\alpha$  is both unital and positive, then it is said to be a *state*. The *state space*  $S(A)$  of  $A$  is the compact convex space of states on  $A$  equipped with the weak- $*$  topology.

If  $B$  is another function system, then a map  $\phi : A \rightarrow B$  is said to be *unital* if  $\phi(1_A) = 1_B$ , and is said to be *positive* if  $\phi(A^+) \subset B^+$ . If  $\phi$  is

bijjective, then it is said to be an *order isomorphism* if it is unital and both  $\phi$  and  $\phi^{-1}$  are positive.

A *concrete function system* is a unital self-adjoint subspace of a unital commutative  $C^*$ -algebra. Observe that a concrete function system, considered as a vector space over the real numbers, is an abstract function system in the above sense. By Kadison's representation theorem, every abstract function system is order isomorphic to a canonical concrete function system. We collect this result, as well as several closely related results on function systems in the next theorem. For details we refer the reader to the book of Alfsen and Shultz [2].

**Theorem 2.2** (Kadison). *Let  $A$  be a function system with state space  $K := S(A)$ . Then  $A$  is order isomorphic to a dense subspace of the space  $A(K)$  of continuous affine function on  $K$  via the map  $\iota : A \rightarrow A(K) : a \rightarrow \hat{a}$ , where*

$$\hat{a}(\alpha) = \alpha(a) \quad \text{for } a \in A, \alpha \in K.$$

*Moreover, if  $A$  is a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ , then there is a  $*$ -homomorphism  $q : C(K) \rightarrow C(X)$  such that  $q \circ \iota$  is the identity on  $A$ .*

Observe that if  $A$  is complete, then Theorem 2.2 implies in particular that  $A$  is order isomorphic to  $A(K)$ . We will work with function systems that are complete in this paper.

Theorem 2.2 largely reduces the study of abstract function systems to the study of the concrete function systems of continuous affine functions on compact convex sets.

**2.3. Choquet boundary.** For an overview of Choquet theory, we refer the reader to the books of Alfsen [1] and Phelps [19].

Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ . Let  $K = S(A)$  denote the state space of  $A$ , and let  $\iota : A \rightarrow A(K)$  and  $q : C(K) \rightarrow C(X)$  be as in Theorem 2.2. Then letting  $q^* : C(X)^* \rightarrow C(K)^*$  denote the adjoint of  $q$  and identifying points in  $X$  and  $K$  with the corresponding point evaluations,  $q^*$  maps  $X$  into  $K$ . The *Choquet boundary*  $\partial_A X$  of  $A$  is  $\partial_A X = (q^*)^{-1}(\partial K)$  where  $\partial K$  denotes the set of extreme points of  $K$ . Note that  $q^*(\partial_A X) = \partial K$ .

In particular, observe that the Choquet boundary  $\partial_{A(K)} K$  of  $A(K)$  is precisely the set  $\partial K$  of extreme points of  $K$ .

**2.4. Choquet order.** Let  $K$  be a compact convex subset of a locally convex vector space and let  $\mu \in M^+(K)$  be a positive measure. If  $K$  is metrizable, then the set  $\partial K$  of extreme points of  $K$  is a  $G_\delta$  set. In this case, we will say that  $\mu$  is *supported* on  $\partial K$  if  $\mu(\partial K) = \mu(K)$ .



If  $K$  is non-metrizable, then Bishop-de Leeuw [8] showed that  $\partial K$  is not necessarily even Borel. In general, we will say that  $\mu$  is *pseudo-supported* on  $\partial K$  if  $\mu(X) = 0$  for every Baire subset  $X \subset K$  with  $X \cap \partial K = \emptyset$ . Recall that a subset of  $K$  is a Baire set if it belongs to the  $\sigma$ -algebra generated by all compact  $G_\delta$  subsets of  $K$ . If  $K$  is metrizable, then every closed subset is a  $G_\delta$ . Thus, in this case,  $\mu$  is pseudo-supported on  $\partial K$  if and only if it is supported on  $\partial K$ . In the general case, one can at least assert that a measure which is pseudo-supported on  $\partial K$  is supported on  $\overline{\partial K}$ .

**Definition 2.3** (Choquet order). Let  $K$  be a compact convex subset of a locally convex vector space. The *Choquet order* “ $\prec_c$ ” on  $M^+(K)$  is defined for  $\mu, \nu \in M^+(K)$  by  $\mu \prec_c \nu$  if  $\mu(f) \leq \nu(f)$  for every continuous convex function  $f \in C(K)$ .

**Definition 2.4** (Boundary measure). Let  $K$  be a compact convex subset of a locally convex vector space. A measure in  $M^+(K)$  is said to be a *boundary measure* if it is maximal in the Choquet order.

The next result combines [1, Prop.I.4.5, Cor.I.4.12].

**Theorem 2.5** (Mokobodzki, Bishop-de Leeuw). *Let  $K$  be a compact convex subset of a locally convex vector space. Then every boundary measure in  $M^+(K)$  is pseudo-supported on  $\partial K$ . If  $K$  is metrizable, then conversely, every measure in  $M^+(K)$  that is supported on  $\partial K$  is a boundary measure.*

For a point  $x \in K$ , let  $\delta_x$  denote the corresponding point mass. The set  $P_x(K) := \{\mu \in P(K) \mid \delta_x \prec_c \mu\}$  is precisely the set of probability measures on  $K$  that represent  $x$ , in the sense that for  $\mu \in P_x(K)$ ,  $\mu(x) = a(x)$  for all  $a \in A(K)$ .

**2.5. Cartier’s theorem.** In classical Choquet theory, there is a notion of *dilation of measures* that, at first glance, appears to have little in common with the notion of dilation arising in the theory of completely positive linear maps on  $C^*$ -algebras. Cartier’s theorem characterizes the Choquet order for metrizable compact convex sets in terms of dilation of measures.

**Theorem 2.6** (Cartier). *Let  $K$  be a metrizable compact convex subset of a locally convex vector space. Let  $\mu, \nu \in M^+(K)$  satisfy  $\mu \prec_c \nu$ . Then  $\mu$  is dilated by  $\nu$ , meaning there is a family  $\{\lambda_x\}_{x \in K} \subset P(K)$  of probability measures such that*

- (1)  $\lambda_x \in P_x(K)$  for  $\mu$ -a.e.  $x \in K$ ,
- (2)  $f \rightarrow \lambda_x(f)$  is  $\mu$ -measurable for all  $f \in C(K)$ , and



$$(3) \int_K f d\nu = \int_K \lambda_x(f) d\mu \text{ for all } f \in C(K).$$

In the metrizable setting, Cartier's theorem can be used to give a short proof of one direction of the equivalence between Choquet order and dilation order. In the non-metrizable setting, where we are unable to make use of Cartier's theorem, we will need to work much harder to prove this result. However, our methods will yield an extension of Cartier's theorem to the non-metrizable setting, which seems to have been an open problem for a considerable time (see e.g. [13]).

### 3. DILATION ORDER

In this section we introduce the dilation order for measures on a compact convex set and prove that it is equivalent to the Choquet order.

**Definition 3.1.** Let  $K$  be a compact convex subset of a locally convex vector space and let  $\mu, \nu \in M^+(K)$  be measures. We say that a representation  $(\pi, H, \xi)$  of  $\mu$  is *dilated* by a representation  $(\sigma, L, \eta)$  of  $\nu$  if there is an isometry  $V : H \rightarrow L$  such that

- (1)  $\pi(a) = V^* \sigma(a) V$  for every  $a \in A(K)$ , and
- (2)  $\eta = V \xi$ .

**Definition 3.2.** Let  $K$  be a compact convex subset of a locally convex vector space. The *dilation order* " $\prec_d$ " on  $M^+(K)$  is defined for  $\mu, \nu \in M^+(K)$  by  $\mu \prec_d \nu$  if some representation of  $\mu$  is dilated by a representation of  $\nu$  in the sense of Definition 3.1.

The next two propositions are useful for working with the dilation order.

**Proposition 3.3.** *Let  $K$  be a compact convex subset of a locally convex vector space and let  $\mu, \nu \in M^+(K)$ . If  $\mu \prec_d \nu$ , then the GNS representation of  $\mu$  is dilated by a representation of  $\nu$ .*

*Proof.* Suppose that  $\mu \prec_d \nu$ . Then some representation  $(\pi, H, \xi)$  of  $\mu$  is dilated by a representation  $(\sigma, L, \eta)$  of  $\nu$ . We claim that the GNS representation  $(\pi_\mu, L^2(\mu), 1_\mu)$  of  $\mu$  is also dilated by  $(\sigma, L, \eta)$ . To see this, let  $V : H \rightarrow L$  be the isometry as in Definition 3.1. By the remarks in Section 2.1, the restriction of  $\pi$  to the cyclic invariant subspace for  $\pi$  generated by  $\xi$  is unitarily equivalent to  $\pi_\mu$  via a unitary that maps  $1_\mu$  to  $\xi$ . Hence we can assume that  $H = L^2(\mu) \oplus H'$  for some Hilbert space  $H'$ ,  $\xi = 1_\mu$  and  $\pi = \pi_\mu \oplus \pi'$  for some  $*$ -representation  $\pi' : C(K) \rightarrow \mathcal{B}(H')$ . Let  $W : L^2(\mu) \rightarrow L$  be the restriction  $W = V|_{L^2(\mu)}$ . Then  $W$  is an isometry,  $\pi_\mu(a) = W^* \sigma(a) W$  and  $\eta = W 1_\mu$ . Hence  $(\pi_\mu, L^2(\mu), 1_\mu)$  is dilated by  $(\sigma, L, \eta)$ .  $\square$

**Proposition 3.4.** *Let  $K$  be a compact convex subset of a locally convex vector space and let  $\mu, \nu \in M^+(K)$ . Suppose there is a positive map  $\Phi : C(K) \rightarrow \mathcal{B}(L^2(\mu))$  such that*

$$\Phi(a) = \pi_\mu(a) \quad \text{for all } a \in A(K),$$

and

$$\nu(f) = \langle \Phi(f)1_\mu, 1_\mu \rangle \quad \text{for all } f \in C(K).$$

Then  $\mu \prec_d \nu$ .

*Proof.* Since  $C(K)$  is commutative, a result of Stinespring (see e.g. [18, Theorem 3.11]) implies that  $\Phi$  is completely positive. Therefore, we may apply Stinespring's dilation theorem (see e.g. [18, Theorem 4.1]) to obtain a  $*$ -representation  $\pi : C(K) \rightarrow \mathcal{B}(L)$  and an isometry  $V : L^2(\mu) \rightarrow L$  such that

$$\Phi(f) = V^* \pi(f) V \quad \text{for all } f \in C(K).$$

In particular,

$$\pi_\mu(a) = V^* \pi(a) V \quad \text{for all } a \in A(K),$$

and setting  $\xi = V1_\mu$ , it follows that for all  $f \in C(K)$ ,

$$\langle \pi(f)\xi, \xi \rangle = \langle V^* \pi(f) V 1_\mu, 1_\mu \rangle = \langle \Phi(f)1_\mu, 1_\mu \rangle = \nu(f).$$

Therefore  $\mu \prec_d \nu$ . □

We will now prove that the dilation order is equivalent to the Choquet order. It will be convenient to handle each direction of the equivalence separately.

**Theorem 3.5.** *Let  $K$  be a compact convex subset of a locally convex vector space, and let  $\mu, \nu \in M^+(K)$ . Then  $\mu \prec_d \nu$  implies that  $\mu \prec_c \nu$ .*

*Proof.* Suppose that  $\mu \prec_d \nu$ . Then by Proposition 3.3, the GNS representation  $(\pi_\mu, L^2(\mu), 1_\mu)$  of  $\mu$  is dilated by a representation  $(\pi, H, \xi)$  of  $\nu$ . Let  $V : L^2(\mu) \rightarrow H$  be an isometry implementing this dilation, i.e. such that  $\pi_\mu(a) = V^* \pi(a) V$  for every  $a \in A(K)$  and  $V1_\mu = \xi$ .

By the remarks in Section 2.1, the restriction of  $\pi$  to the cyclic invariant subspace generated by  $\xi$  is unitarily equivalent to  $\pi_\nu$  via a unitary that maps  $1_\nu$  to  $\xi$ . Hence we can assume that  $H = L^2(\nu) \oplus H'$  for some Hilbert space  $H'$ ,  $\xi = 1_\nu$  and  $\pi = \pi_\nu \oplus \pi'$  for some  $*$ -representation  $\pi' : C(K) \rightarrow \mathcal{B}(H')$ . Note that  $VL^2(\mu)$  is not necessarily contained in  $L^2(\nu)$ . Let  $P_\nu \in \mathcal{B}(H)$  denote the orthogonal projection onto  $L^2(\nu)$ .

Fix a continuous convex function  $f \in C(K)$ . We must show that  $\mu(f) \leq \nu(f)$ . By continuity and compactness, for every  $\epsilon > 0$  there

are finitely many closed convex subsets  $\{K_i\}_{i=1}^n$  in  $K$  such that  $K = \bigcup_{1 \leq i \leq n} K_i$  and for  $1 \leq i \leq n$ ,

$$|f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in K_i.$$

For  $1 \leq j \leq n$ , let  $K'_j = K_j \setminus \bigcup_{1 \leq i < j} K_i$ . Define functions  $h_j = \chi_{K'_j}$  in  $L^2(\mu)$  and measures  $\mu_j = h_j d\mu$  on  $K$ . Then  $\sum_{j=1}^n h_j = 1_\mu$  and hence  $\sum_{j=1}^n \mu_j = \mu$ .

Define functions  $k_j = P_\nu V h_j \in L^2(\nu)$  and measures  $\nu_j = k_j d\nu$  on  $K$ . Then

$$\sum_{j=1}^n k_j = \sum_{j=1}^n P_\nu V \sum_{j=1}^n h_j = P_\nu V 1_\mu = P_\nu \xi = 1_\nu,$$

and hence  $\sum_{j=1}^n \nu_j = \nu$ .

Let  $x_j$  denote the barycenter of  $\mu_j$ . Then  $x_j \in K_j$ . We claim that  $x_j$  is also the barycenter of  $\nu_j$ , or equivalently that  $\mu_j$  and  $\nu_j$  agree on  $A(K)$ . To see this, we compute for  $a \in A(K)$ ,

$$\begin{aligned} \mu_j(a) &= \int_K a d\mu_j = \langle \pi_\mu(a) h_j, 1_\mu \rangle \\ &= \langle V^* \pi(a) V h_j, 1_\mu \rangle = \langle \pi(a) V h_j, V 1_\mu \rangle \\ &= \langle P_\nu \pi(a) V h_j, \xi \rangle = \langle P_\nu \pi(a) V h_j, 1_\nu \rangle \\ &= \langle \pi_\nu(a) P_\nu V h_j, 1_\nu \rangle = \langle \pi_\nu(a) k_j, 1_\nu \rangle \\ &= \int_K a k_j d\nu = \int_K a d\nu_j \\ &= \nu_j(a). \end{aligned}$$

Furthermore, we obtain  $\mu_j(K) = \mu_j(1_\mu) = \nu_j(1_\nu) = \nu_j(K)$ .

Since  $\nu_j$  has barycenter  $x_j$ , the convexity of  $f$  implies that

$$f(x_j) \leq \frac{\nu_j(f)}{\nu_j(K)}.$$

Also, by construction,

$$\begin{aligned} \left| \frac{\mu_j(f)}{\mu_j(K)} - f(x_j) \right| &= \left| \frac{1}{\mu_j(K)} \int_K f d\mu_j - f(x_j) \right| \\ &= \left| \frac{1}{\mu_j(K)} \int_K (f(x) - f(x_j)) d\mu_j(x) \right| \\ &\leq \frac{1}{\mu_j(K)} \int_{K_j} |f(x) - f(x_j)| d\mu_j(x) \\ &\leq \epsilon, \end{aligned}$$

where we have used the fact that  $\mu_j$  is supported on  $K'_j$ . Hence

$$\mu_j(f) \leq \mu_j(K)(f(x_j) + \epsilon) = \nu_j(K)(f(x_j) + \epsilon).$$

Now we estimate

$$\begin{aligned} \mu(f) &= \sum_{j=1}^n \mu_j(f) \leq \sum_{j=1}^n \nu_j(K)(f(x_j) + \epsilon) \\ &\leq \sum_{j=1}^n \nu_j(K) \left( \frac{\nu_j(f)}{\nu_j(K)} + \epsilon \right) \\ &= \sum_{j=1}^n \nu_j(f) + \epsilon \sum_{j=1}^n \nu_j(K) \\ &= \nu(f) + \epsilon. \end{aligned}$$

Since  $\epsilon$  was arbitrary, it follows that  $\mu(f) \leq \nu(f)$ . Hence  $\mu \prec_c \nu$ .  $\square$

For the other equivalence, we will first give a short proof in the metrizable case using Cartier's theorem from Section 2.5. The apparent simplicity of the proof is deceptive, since the proof of Cartier's theorem requires a significant amount of work using the theory of disintegration of measures.

**Theorem 3.6.** *Let  $K$  be a metrizable compact convex subset of a locally convex space, and let  $\mu, \nu \in M^+(K)$ . Then  $\mu \prec_c \nu$  implies  $\mu \prec_d \nu$ .*

*Proof.* By Theorem 2.6 (Cartier's theorem), there is a family of probability measures  $\{\lambda_x : x \in K\}$  such that  $\delta_x \prec_c \lambda_x$  for  $\mu$ -a.e.  $x \in K$ , the map  $f \rightarrow \lambda_x(f)$  is  $\mu$ -measurable and  $\nu(f) = \int_K \lambda_x(f) d\mu$  for all  $f \in C(K)$ .

Define a positive unital map  $\Phi : C(K) \rightarrow L^\infty(\mu)$  by  $\Phi(f)(x) = \lambda_x(f)$ , and identify functions in  $L^\infty(\mu)$  with the corresponding multiplication operators in  $\mathcal{B}(L^2(\mu))$ . Then for  $a \in A(K)$ , we have  $\Phi(a)(x) = a(x)$   $\mu$ -a.e., giving  $\Phi(a) = \pi_\mu(a)$ .

Now for  $f \in C(K)$ ,

$$\langle \Phi(f)1_\mu, 1_\mu \rangle = \int_K \lambda_x(f) d\mu = \nu(f).$$

Therefore  $\mu \prec_d \nu$  by Proposition 3.4.  $\square$

We now present a proof that is also valid in the non-metrizable case, where Cartier's theorem does not apply. The proof utilizes operator-algebraic methods, and does not require the theory of disintegration of measures. In fact, as a consequence of our approach, we will obtain an extension of Cartier's theorem that is valid in the non-metrizable case.

We begin with a lemma which is readily obtained by taking a direct sum of two representations.

**Lemma 3.7.** *Let  $K$  be a compact convex subset of a locally convex vector space, and let  $\mu_1, \mu_2, \nu_1, \nu_2 \in M^+(K)$  satisfy  $\mu_1 \prec_d \nu_1$  and  $\mu_2 \prec_d \nu_2$ . Then  $\mu_1 + \mu_2 \prec_d \nu_1 + \nu_2$ .*

We first handle the case of an atomic measure.

**Proposition 3.8.** *Let  $K$  be a compact convex subset of a locally convex vector space, and let  $\mu, \nu \in M^+(K)$ . Suppose that  $\mu$  is atomic, and thus that it has finite or countable support. Then  $\mu \prec_c \nu$  implies  $\mu \prec_d \nu$ .*

*Proof.* Since  $\mu$  is atomic, we can write it as  $\mu = \sum_{i=1}^{\kappa} \alpha_i \delta_{x_i}$  for positive real numbers  $(\alpha_i)_{i=1}^{\kappa}$  summing to one and points  $(x_i)_{i=1}^{\kappa}$  in  $K$ , where  $\kappa \in \mathbb{N} \cup \{\infty\}$ . By repeated use of the Cartier-Fell-Meyer theorem [1, Proposition I.3.2], we obtain measures  $(\nu_i)_{i=1}^{\kappa}$  in  $P(K)$  such that

$$\delta_{x_i} \prec_c \nu_i \quad \text{and} \quad \nu = \sum_{i=1}^{\kappa} \alpha_i \nu_i.$$

The GNS representation  $\pi_{\mu} : C(K) \rightarrow B(L^2(\mu))$  of  $\mu$  can be written as

$$\pi_{\mu}(f) = \sum_{i=1}^{\kappa} f(x_i) \chi_{x_i} \quad \text{for } f \in C(K).$$

Define  $\Phi : C(K) \rightarrow B(L^2(\mu))$  by

$$\Phi(f) = \sum_{i=1}^{\kappa} \nu_i(f) \chi_{x_i} \quad \text{for } f \in C(K).$$

Then  $\Phi(a) = \pi_{\mu}(a)$  for all  $a \in A(K)$ , and an easy computation shows

$$\langle \Phi(f) 1_{\mu}, 1_{\mu} \rangle = \sum_{i=1}^{\kappa} \alpha_i \nu_i(f) = \nu(f).$$

By Proposition 3.4,  $\mu \prec_d \nu$ . □

If  $K$  is a compact convex subset of a locally convex vector space, and  $\mu, \nu \in M^+(K)$  satisfy  $\mu \prec_c \nu$ , then we can decompose  $\mu = \mu_d + \mu_c$  into its atomic and continuous parts. The Cartier-Fell-Meyer theorem [1, Proposition I.3.2] implies that we can decompose  $\nu$  as  $\nu = \nu_1 + \nu_2$  for  $\nu_1, \nu_2 \in M^+(K)$  satisfying  $\mu_d \prec_c \nu_1$  and  $\mu_c \prec_c \nu_2$ . Proposition 3.8 yields  $\mu_d \prec_d \nu_1$ . Thus by Lemma 3.7, it remains to deal with the continuous part. This is accomplished by approximating  $\mu_c$  in an appropriate way by atomic measures.

Consider the following set of pairs of probability measures that are comparable in the Choquet order:

$$M_c := \{(\mu, \nu) : \mu \prec_c \nu, \mu, \nu \in P(K)\}.$$

By [1, Lemma I.3.7],  $M_c$  is a weak-\* compact convex subset of the product  $P(K) \times P(K)$ , and the extreme points of  $M_c$  are contained in  $S = \bigcup_{x \in K} \{\delta_x\} \times P_x(K)$ . In particular, it follows from the Krein-Milman theorem that the convex hull of  $S$  is weak-\* dense in  $M_c$ .

We require the following technical approximation result.

**Lemma 3.9.** *Let  $\mu, \nu \in P(K)$  be probability measures such that  $\mu$  is continuous and  $\mu \prec_c \nu$ . Let  $E \subset A(K)$ ,  $F \subset C(K)$  and  $\Xi \subset L^2(\mu)$  be finite sets. Then for  $\varepsilon > 0$ , there is a unital positive map  $\Phi : C(K) \rightarrow \mathcal{B}(L^2(\mu))$  satisfying*

$$|\langle \Phi(a)\xi, \eta \rangle - \langle \pi_\mu(a)\xi, \eta \rangle| < \varepsilon \quad \text{for all } a \in E \text{ and all } \xi, \eta \in \Xi,$$

and

$$|\langle \Phi(f)1_\mu, 1_\mu \rangle - \nu(f)| < \varepsilon \quad \text{for all } f \in F.$$

*Proof.* Let  $N = \max\{1, \|a\|_\infty, \|\xi\|^2 : a \in E, \xi \in \Xi\}$ , and set  $\varepsilon' = \varepsilon/2N$ . Choose  $\delta > 0$  so that if  $\mu(C) < \delta$ , then

$$\max\{\|\chi_C \xi\|^2 : \xi \in \Xi\} < \frac{\varepsilon'}{2} = \frac{\varepsilon}{4N}.$$

By the uniform continuity of  $a \in E$  and the compactness of  $K$ , we may find a finite collection of compact convex sets  $K_i \subset K$  for  $1 \leq i \leq n$  such that their interiors cover  $K$  and

$$|a(x) - a(y)| < \varepsilon' \quad \text{for all } a \in E \text{ and } x, y \in K_i, 1 \leq i \leq n.$$

Choose Borel sets  $B_i \subset \overline{B_i} \subset \text{int}(K_i)$  that partition  $K$  into a disjoint union of sets. Define  $\mu_i = \mu|_{B_i}$ . Applying the Cartier-Fell-Meyer theorem [1, Proposition I.3.2] yields measures  $\nu_i$  such that  $\mu_i \prec_c \nu_i$  and  $\sum_{i=1}^n \nu_i = \nu$ . By Urysohn's lemma, there are functions  $g_i$  in  $C(K)$  such that  $\chi_{B_i} \leq g_i \leq \chi_{K_i}$ .

Applying [1, Lemma I.3.7] as mentioned above to  $(\mu_i, \nu_i)$ , we can find positive measures  $(\sigma_i, \tau_i)$  in the convex hull of  $S$  such that  $\sigma_i$  is finitely supported and the pair  $(\sigma_i, \tau_i)$  approximates  $(\mu_i, \nu_i)$ . Specifically, there are constants  $\alpha_{ik} > 0$ , points  $x_{ik} \in K$  and probability measures  $\tau_{ij} \in P_{x_{ij}}(K)$  such that

$$\sum_{k=1}^{m_i} \alpha_{ik} = \|\mu_i\|, \quad \sigma_i = \sum_{k=1}^{m_i} \alpha_{ik} \delta_{x_{ik}}, \quad \tau_i = \sum_{j=1}^{m_i} \alpha_{ij} \tau_{ij},$$

and such that  $\sigma_i$  approximates  $\mu_i$  and  $\tau_i$  approximates  $\nu_i$ , meaning

$$|\sigma_i(g_i) - \mu_i(g_i)| < \delta \|\mu_i\| \quad \text{for } 1 \leq i \leq n,$$

and

$$|\tau_i(f) - \nu_i(f)| < \varepsilon \|\mu_i\| \quad \text{for all } f \in F \text{ and } 1 \leq i \leq n.$$

Notice that

$$\|\mu_i\| = \|\nu_i\| = \|\sigma_i\| = \|\tau_i\| = \sum_{k=1}^{m_i} \alpha_{ik}.$$

The condition

$$\delta \|\mu_i\| > |\sigma_i(g_i) - \mu_i(g_i)| = \|\mu_i\| - \sum_{k=1}^{m_i} \alpha_{ik} g(x_{ik})$$

shows that the set  $\mathcal{G}_i = \{k : x_{ik} \in K_i\}$  is sufficiently large that

$$\sum_{k \in \mathcal{G}_i} \alpha_{ik} > (1 - \delta) \|\mu_i\|.$$

This means that most of the mass of  $\sigma_i$  is supported on  $K_i$ .

Partition each  $B_i$  into disjoint Borel sets  $B_{ik}$  so that  $\mu_i(B_{ik}) = \alpha_{ik}$ . Define  $C = \bigcup_{i=1}^n \bigcup_{k \notin \mathcal{G}_i} B_{ik}$ . Then

$$\mu(C) = \sum_{i=1}^n \sum_{k \notin \mathcal{G}_i} \alpha_{ik} < \delta \sum_{i=1}^n \|\mu_i\| = \delta.$$

Define a positive map  $\Phi : C(K) \rightarrow L^\infty(\mu)$  by

$$\Phi(f) = \sum_{i=1}^n \sum_{k=1}^{m_i} \tau_{ik}(f) \chi_{B_{ik}} \quad \text{for } f \in C(K).$$

Identify  $L^\infty(\mu)$  with the corresponding multiplication operators on  $\mathcal{B}(L^2(\mu))$ , so that the range of  $\Phi$  is contained in  $\mathcal{B}(L^2(\mu))$ . Evidently  $\Phi$  is positive, and since each  $\tau_{ij}$  is a probability measure, it is clear that  $\Phi(1) = 1$ .

To verify the second inequality, we compute

$$\begin{aligned} \langle \Phi(f) 1_\mu, 1_\mu \rangle &= \sum_{i=1}^n \sum_{k=1}^{m_i} \tau_{ik}(f) \int_K \chi_{B_{ik}} d\mu \\ &= \sum_{i=1}^n \sum_{k=1}^{m_i} \alpha_{ik} \tau_{ik}(f) \\ &= \sum_{j=1}^n \tau_j(f). \end{aligned}$$



Therefore for  $f \in F$ , we have

$$\begin{aligned} |\langle \Phi(f)1_\mu, 1_\mu \rangle - \nu(f)| &= \left| \sum_{j=1}^n \tau_j(f) - \sum_{j=1}^n \nu_j(f) \right| \\ &\leq \sum_{i=1}^n |\tau_i(f) - \nu_i(f)| \\ &< \sum_{i=1}^n \varepsilon \|\mu_i\| = \varepsilon. \end{aligned}$$

To verify the first inequality, first observe that for  $a \in E$  and  $\xi, \eta \in \Xi$ ,

$$\begin{aligned} \langle \Phi(a)\xi, \eta \rangle &= \sum_{i=1}^n \sum_{k=1}^{m_i} \tau_{ik}(a) \langle \chi_{B_{ik}} \xi, \eta \rangle \\ &= \sum_{i=1}^n \sum_{k=1}^{m_i} a(x_{ik}) \langle \chi_{B_{ik}} \xi, \eta \rangle \end{aligned}$$

and

$$\langle \pi_\mu(a)\xi, \eta \rangle = \sum_{i=1}^n \sum_{k=1}^{m_i} \langle a \chi_{B_{ik}} \xi, \eta \rangle.$$

Recall that  $|a(x) - a(x_{ik})| < \varepsilon'$  if  $x \in K_i$  and  $k \in \mathcal{G}_i$  (so that  $x_{ik} \in K_i$ ). Otherwise  $|a(x) - a(x_{ik})| \leq 2\|a\|_\infty \leq 2N$ . Also, since  $\mu(C) < \delta$ , we have  $\|\chi_C \xi\|^2 < \varepsilon'/2 = \varepsilon/4N$ . Thus we obtain

$$\begin{aligned} |\langle (\Phi(a) - \pi_\mu(a))\xi, \eta \rangle| &\leq \sum_{i=1}^n \sum_{k=1}^{m_i} |\langle (a - a(x_{ik})) \chi_{B_{ik}} \xi, \eta \rangle| \\ &\leq \sum_{i=1}^n \sum_{k \in \mathcal{G}_i} \varepsilon' \|\chi_{B_{ik}} \xi\| \|\chi_{B_{ik}} \eta\| \\ &\quad + \sum_{i=1}^n \sum_{k \notin \mathcal{G}_i} 2\|a\|_\infty \|\chi_{B_{ik}} \xi\| \|\chi_{B_{ik}} \eta\| \\ &\leq \frac{\varepsilon}{2N} \left( \sum_{i,k} \|\chi_{B_{ik}} \xi\|^2 \right)^{1/2} \left( \sum_{i,k} \|\chi_{B_{ik}} \eta\|^2 \right)^{1/2} \\ &\quad + 2N \left( \sum_{i,k \notin \mathcal{G}_i} \|\chi_{B_{ik}} \xi\|^2 \right)^{1/2} \left( \sum_{i,k \notin \mathcal{G}_i} \|\chi_{B_{ik}} \eta\|^2 \right)^{1/2} \\ &< \frac{\varepsilon}{2N} \|\xi\| \|\eta\| + 2N \|\chi_C \xi\| \|\chi_C \eta\| \end{aligned}$$

$$< \frac{\varepsilon}{2} + 2N \frac{\varepsilon}{4N} = \varepsilon.$$

This completes the proof of the lemma.  $\square$

We are now ready to prove the final piece of the equivalence between Choquet order and dilation order.

**Theorem 3.10.** *Let  $K$  be a compact convex subset of a locally convex vector space, and let  $\mu, \nu \in M^+(K)$ . Then  $\mu \prec_c \nu$  implies that  $\mu \prec_d \nu$ .*

*Proof.* First assume that  $\mu$  is a continuous measure. Form a net  $\Lambda$  consisting of tuples  $(E, F, \Xi, \varepsilon)$  where  $E \subset A(K)$ ,  $F \subset C(K)$  and  $\Xi \subset L^2(\mu)$  are finite sets, and  $\varepsilon > 0$ . Say  $(E, F, \Xi, \varepsilon) \leq (E', F', \Xi', \varepsilon')$  provided that  $E \subset E'$ ,  $F \subset F'$ ,  $\Xi \subset \Xi'$  and  $\varepsilon > \varepsilon'$ . For each  $\lambda \in \Lambda$ , define a positive map  $\Phi_\lambda$  using Lemma 3.9.

The net  $\{\Phi_\lambda\}_{\lambda \in \Lambda}$  is bounded by 1 since each  $\Phi_\lambda$  is unital and positive. By passing to a cofinal subnet, we obtain a point-WOT limit  $\Phi$  which will also be a unital positive map from  $C(K)$  into  $\mathcal{B}(L^2(\mu))$ . By the properties of  $\Phi_\lambda$  obtained from Lemma 3.9, we see that for all  $a \in A(K)$  and  $\xi, \eta \in L^2(\mu)$ ,

$$\langle \Phi(a)\xi, \eta \rangle = \lim_{\lambda} \langle \Phi_\lambda(a)\xi, \eta \rangle = \langle \pi_\mu(a)\xi, \eta \rangle.$$

Furthermore, for all  $f \in C(K)$ ,

$$\langle \Phi(f)1_\mu, 1_\mu \rangle = \lim_{\lambda} \langle \Phi_\lambda(f)1_\mu, 1_\mu \rangle = \nu(f).$$

It follows that  $\Phi$  satisfies the hypotheses of Proposition 3.4. Therefore  $\mu \prec_d \nu$ .

For a general measure  $\mu$ , we decompose  $\mu$  as  $\mu = \mu_d + \mu_c$  where  $\mu_d$  is the discrete (atomic) part of  $\mu$  and  $\mu_c$  is the continuous part. As indicated before Lemma 3.9, we use the Cartier-Fell-Meyer theorem to decompose  $\nu$  as  $\nu = \nu_1 + \nu_2$  with  $\mu_d \prec_c \nu_1$  and  $\mu_c \prec_c \nu_2$ . Applying Proposition 3.8 to the first pair of measures and the previous paragraph to the second pair, we obtain  $\mu_d \prec_d \nu_1$  and  $\mu_c \prec_d \nu_2$ . The result now follows from Lemma 3.7.  $\square$

As a consequence of the proofs in this section, we also have the following alternative description of the dilation order.

**Corollary 3.11.** *Let  $K$  be a compact convex subset of a locally compact vector space, and let  $\mu, \nu \in M^+(K)$ . Then  $\mu \prec_d \nu$  if and only if there is a unital positive map  $\Phi : C(K) \rightarrow L^\infty(\mu)$  such that*

- (1)  $\Phi(a) = a$  for all  $a \in A(K)$ , and
- (2)  $\nu(f) = \int_K \Phi(f) d\mu$  for all  $f \in C(K)$ .

#### 4. CONSEQUENCES FOR CHOQUET THEORY

It is a major result in the metrizable setting due to Mokobodzki [1, Proposition I.4.5 and (4.11) p.35] that the maximal elements in the Choquet order are precisely the measures which are supported on the set of extreme points. The following corollary is an immediate consequence of Mokobodzki's result and the results in Section 3.

**Corollary 4.1.** *Let  $K$  be a metrizable compact convex subset of a locally convex vector space. The following are equivalent for  $\mu \in M^+(K)$ :*

- (1)  $\mu$  is a boundary measure (i.e.  $\mu$  is maximal in Choquet order),
- (2)  $\mu$  is maximal in the dilation order,
- (3)  $\mu$  is supported on  $\partial K$ .

Bishop and de Leeuw observed that if  $K$  is a non-metrizable compact convex subset of a locally convex vector space, then the set  $\partial K$  of extreme points is not necessarily even Borel. However, they showed that if  $\mu \in M^+(K)$  is a boundary measure, then  $\mu$  is pseudo-supported on  $\partial K$ , i.e.,  $\mu(C) = 0$  for every Baire set that is disjoint from  $\partial K$ .

It can happen that a measure pseudo-supported on  $\partial K$  is not a boundary measure, as shown by an example of Mokobodzki [19, p.60]. However, things are better if  $\partial K$  is Borel and  $\mu$  is supported on  $\partial K$  in the usual sense. In this case, every measure supported on  $\partial K$  is a boundary measure [19, Corollary 10.8]. Conversely, every boundary measure is supported on the Shilov boundary  $\overline{\partial K}$  [1, Proposition I.4.6]. Thus, when  $\partial K$  is closed, we obtain the following corollary which applies in the general, possibly non-metrizable setting.

**Corollary 4.2.** *Let  $K$  be a compact convex subset of a locally convex vector space such that the set  $\partial K$  of extreme points is closed. The following are equivalent for  $\mu \in M^+(K)$ :*

- (1)  $\mu$  is a boundary measure (i.e.  $\mu$  is maximal in Choquet order),
- (2)  $\mu$  is maximal in the dilation order,
- (3)  $\mu$  is supported on  $\partial K$ .

Now we will explain how Cartier's theorem can be extended to the non-metrizable case. The major difficulty is that Cartier's proof requires the theory of disintegration of measures. We will instead utilize Corollary 3.11, which we obtained as a consequence of our proof that the dilation order is equivalent to the Choquet order.

The other ingredient is a lifting theorem of Maharam [17] (see [21] or [22] for an alternate proof). Let  $(X, \mathcal{B}, \mu)$  be a complete probability space, and let  $M^\infty(\mathcal{B})$  denote the space of bounded measurable functions on  $X$ . There is a natural quotient map  $q : M^\infty(\mathcal{B}) \rightarrow L^\infty(\mu)$

obtained by identifying functions which agree  $\mu$ -a.e. Using  $C^*$ -algebraic terminology, Maharam's lifting theorem says that there is a unital  $*$ -monomorphism  $\rho : L^\infty(\mu) \rightarrow M^\infty(\mathcal{B})$  such that  $q \circ \rho$  is the identity map on  $L^\infty(\mu)$ . In other words, there is a positive unital lifting of this quotient map which is also multiplicative. We actually only require the positivity of this lifting, and not the fact that it is a homomorphism.

**Theorem 4.3.** *Let  $K$  be a compact convex subset of a locally convex vector space. Let  $\mu, \nu \in M^+(K)$  satisfy  $\mu \prec_c \nu$ . Then there is a family  $\{\lambda_x\}_{x \in K} \subset P(K)$  of probability measures such that*

- (1)  $\lambda_x(a) = a(x)$   $\mu$ -a.e. for all  $a \in A(K)$ ,
- (2)  $f \rightarrow \lambda_x(f)$  is  $\mu$ -measurable for all  $f \in C(K)$ , and
- (3)  $\int f d\nu = \int \lambda_x(f) d\mu$  for all  $f \in C(K)$ .

*Proof.* By Theorem 3.10, we have  $\mu \prec_d \nu$ . Corollary 3.11 yields a unital completely positive map such that  $\Phi : C(K) \rightarrow L^\infty(\mu)$  such that  $\Phi(a) = a$  for  $a \in A(K)$  and  $\nu(f) = \int_K \Phi(f) d\mu$  for all  $f \in C(K)$ . Let  $\rho : L^\infty(\mu) \rightarrow M^\infty(\mathcal{B})$  be a positive unital lifting to the bounded measurable functions on  $K$ . Then for each  $x \in K$ , the positive linear map  $\phi_x : C(K) \rightarrow \mathbb{C}$  defined by

$$\phi_x(f) = \rho \circ \Phi(f)(x) \quad \text{for } f \in C(K)$$

satisfies  $\phi_x(1) = \rho \circ \Phi(1)(x) = 1$ . So this is a state on  $C(K)$ . By the Riesz-Markov-Kakutani representation theorem, there is a regular Borel probability measure  $\lambda_x \in P(K)$  so that  $\phi_x(f) = \int f d\lambda_x$ . Therefore  $f \rightarrow \lambda_x(f) = \rho \circ \Phi(f)(x)$  is measurable for every  $f \in C(K)$ , and

$$\int f d\nu = \int \Phi(f) d\mu = \int \phi_x(f) d\mu(x) = \int \lambda_x(f) d\mu(x).$$

Finally for  $a \in A(K)$ , since  $\Phi(a) = a$ , we have  $\rho(a) = a$   $\mu$ -a.e., and therefore  $\lambda_x(a) = a(x)$   $\mu$ -a.e.  $\square$

**Remark 4.4.** Theorem 4.3 is not quite ideal, because in general we do not know that  $\lambda_x \in P_x(K)$  for  $\mu$ -a.e.  $x \in K$ . The problem is that we do not know  $\rho$  can be chosen to satisfy  $\rho(a) = a$  for all  $a \in C(K)$ . Such a lifting is called a strong lifting. While strong liftings exist in the metrizable case and for certain product measures, in general the existence of a strong lifting is a major open problem (see e.g. [22]). However if  $\rho$  is a strong lifting, then  $\phi_x(a) = \rho \circ \Phi(a)(x) = a(x)$  for all  $a \in A(K)$ , and therefore  $\lambda_x \in P_x(K)$ .

Without a strong lifting, we know that  $\lambda_x(a) = a(x)$  for  $\mu$ -a.e.  $x \in K$  for each  $a \in A(K)$ . If  $K$  is metrizable, then  $A(K)$  is separable, and hence it has a countable spanning set, from which we can deduce that  $\lambda_x \in P_x$   $\mu$ -a.e.  $x \in K$ . This map can then be repaired on a  $\mu$ -null

set to yield a strong lifting. Thus we recover Cartier's theorem in the metrizable case.

## 5. THE UNIQUE EXTENSION PROPERTY

In this section we will relate our dilation-theoretic characterization of Choquet order to some constructions in the theory of operator algebras.

Many of the results from previous sections in this paper can be stated using only classical notions, and in particular do not require the theory of completely positive maps and noncommutative  $C^*$ -algebras. However, at this point we need to shift our focus somewhat.

Recall that if  $A$  is a function system, then a map  $\phi : A \rightarrow \mathcal{B}(H)$  is *completely positive* if each of the maps  $\phi^{(n)} : \mathcal{M}_n(A) \rightarrow \mathcal{M}_n(\mathcal{B}(H))$  defined by

$$\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})]$$

are positive for all  $n \geq 1$ . For the basic theory of completely positive maps, we refer the reader to Paulsen's book [18].

Stinespring proved (see e.g. [18, Theorem 3.11]) that if  $A$  is a commutative  $C^*$ -algebra, then every positive map  $\phi : A \rightarrow \mathcal{B}(H)$  is automatically completely positive. However, this is not true for an arbitrary function system (see e.g. [18, Example 2.2]).

It is an important fact that  $\mathcal{B}(H)$  is injective in the category of operator systems and completely positive maps [4] (see [18, Theorem 7.5]). Thus it follows that every completely positive map  $\phi : A \rightarrow \mathcal{B}(H)$  has a completely positive extension to  $C := C^*(A)$ . On the other hand, if  $\phi$  is positive but not completely positive, then it does not extend to a positive map on  $C$ , because such an extension would be completely positive, which would imply the complete positivity of the original map.

**Definition 5.1.** Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ . A unital completely positive map  $\phi : A \rightarrow \mathcal{B}(H)$  is said to have the *unique extension property* if it has a unique completely positive extension  $\pi : C \rightarrow \mathcal{B}(H)$  and  $\pi$  is a  $*$ -homomorphism. Similarly, we will say that a  $*$ -representation  $\pi : C \rightarrow \mathcal{B}(H)$  has the *unique extension property relative to  $A$*  if the restriction  $\phi = \pi|_A$  has the unique extension property.

**Definition 5.2.** Let  $A$  be a function system and let  $\phi : A \rightarrow \mathcal{B}(H)$  and  $\psi : A \rightarrow \mathcal{B}(L)$  be unital completely positive maps. We say that  $\phi$  is *dilated* by  $\psi$  or that  $\psi$  is a *dilation* of  $\phi$ , and write  $\phi \prec \psi$ , if there is an isometry  $V : H \rightarrow L$  such that  $\phi(a) = V^*\psi(a)V$  for all  $a \in A$ . If, in addition, the subspace  $VH$  is invariant (and hence reducing) for

$\psi(A)$ , then  $\psi$  is said to be a *trivial dilation* of  $\phi$ . If every dilation of  $\phi$  is trivial, then  $\phi$  is said to be *maximal*.

The above notions naturally extend to the more general setting of operator systems, which are unital self-adjoint subspaces of possibly noncommutative  $C^*$ -algebras. In the general setting, completely positive maps are essential since positive maps, even on  $C^*$ -algebras, need not be completely positive.

The next result was established for (generally non-self-adjoint) operator algebras by Dritschel and McCullough [12], and for operator systems by Arveson [6], both in the non-commutative setting.

**Proposition 5.3** (Dritschel-McCullough, Arveson). *Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ . A  $*$ -representation  $\pi : C(X) \rightarrow \mathcal{B}(H)$  has the unique extension property relative to  $A$  if and only if the restriction  $\pi|_A$  is maximal.*

They also establish that maximal dilations always exist, again in the possibly non-commutative setting [6, 12].

**Proposition 5.4** (Dritschel-McCullough, Arveson). *Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ , and let  $\phi : A \rightarrow \mathcal{B}(H)$  be a completely positive unital map. Then there is a maximal completely positive map  $\psi : A \rightarrow \mathcal{B}(L)$  which dilates  $\phi$ .*

We now make a crucial connection between boundary measures and the unique extension property.

**Theorem 5.5.** *Let  $K$  be a compact convex subset of a locally convex vector space, and let  $\mu \in M^+(K)$  be a measure with corresponding GNS representation  $\pi_\mu : C(K) \rightarrow \mathcal{B}(L^2(\mu))$ . Then  $\mu$  is a boundary measure if and only if  $\pi_\mu$  has the unique extension property relative to  $A(K)$ .*

*Proof.* ( $\Rightarrow$ ) Suppose that  $\mu$  is a boundary measure. This means that  $\mu$  is maximal in the Choquet order, and hence by Corollary 4.1, it is also maximal in the dilation order. We must show that if  $\psi : C(K) \rightarrow \mathcal{B}(L^2(\mu))$  is a (completely) positive extension of the restriction  $\pi_\mu|_{A(K)}$ , then  $\psi = \pi_\mu$ .

By Proposition 5.3, it suffices to show that  $\pi_\mu|_{A(K)}$  is maximal in the sense of Definition 5.2. By Proposition 5.4, there is a maximal unital completely positive map  $\phi : A(K) \rightarrow \mathcal{B}(L)$  on a Hilbert space  $L$  that dilates  $\pi_\mu|_{A(K)}$ . Then there is an isometry  $V : H \rightarrow L$  such that  $\pi_\mu(a) = V^*\phi(a)V$  for all  $a \in A(K)$ . By Proposition 5.3,  $\phi$  extends to a  $*$ -representation  $\pi : C(K) \rightarrow \mathcal{B}(L)$ .

Let  $\xi = V1_\mu$ , and define  $\nu \in P(K)$  by

$$\nu(f) = \langle \pi(f)\xi, \xi \rangle \quad \text{for } f \in C(K).$$

Then  $(\pi, H, \xi)$  is a representation of  $\nu$ . Moreover,

$$\pi_\mu(a) = V^*\phi(a)V = V^*\pi(a)V \quad \text{for all } a \in A(K).$$

Hence the GNS representation  $(\pi_\mu, L^2(\mu), 1_\mu)$  is dilated by  $(\pi, H, \xi)$ , implying  $\mu \prec_d \nu$ . By the maximality of  $\mu$  it follows that  $\mu = \nu$ .

By the remarks in Section 2.1, the restriction of  $\pi$  to the cyclic invariant subspace for  $\pi$  generated by  $\xi$  is unitarily equivalent to  $\pi_\mu$ . In other words, the restriction  $\pi_\mu|_{A(K)}$  is unitarily equivalent to a summand of  $\phi$ . Since  $\phi$  is maximal, it follows that  $\pi_\mu|_{A(K)}$  is necessarily maximal.

( $\Leftarrow$ ) Suppose that the restriction  $\pi_\mu|_{A(K)}$  has the unique extension property. Then by Proposition 5.3,  $\pi_\mu|_{A(K)}$  is maximal in the sense of Definition 5.2.

Let  $\nu \in P(K)$  be a probability measure such that  $\mu \prec_d \nu$ . Then by Proposition 3.3, there is a representation  $(\pi, H, \xi)$  of  $\nu$  that dilates the GNS representation  $(\pi_\mu, L^2(\mu), 1_\mu)$ . Let  $V : L^2(\mu) \rightarrow H$  be an isometry implementing this dilation, i.e. such that  $\pi_\mu(a) = V^*\pi(a)V$  for every  $a \in A(K)$  and  $V1_\mu = \xi$ .

The restriction  $\pi_\mu|_{A(K)}$  is dilated by the restriction  $\pi|_{A(K)}$ . Hence, by the maximality of  $\pi_\mu|_{A(K)}$ , the subspace  $VH$  is invariant for  $\pi$ . It follows that the restriction of  $\pi$  to  $VH$  is unitarily equivalent to  $\pi_\mu$  via a unitary that maps  $1_\mu$  to  $\xi$ . Therefore,  $\mu = \nu$  and  $\mu$  is maximal in the dilation order. By Corollary 4.1,  $\mu$  is a boundary measure.  $\square$

In the metrizable case and in the case of a closed set of extreme points, Theorem 5.5 yields a very simple characterization of maps with the unique extension property. There are two other ingredients: the equivalence between boundary measures and measures supported on the extreme boundary obtained in Corollary 4.1 and Corollary 4.2, and the fact that every  $*$ -representation decomposes into a direct sum of cyclic representations.

**Corollary 5.6.** *Let  $K$  be a compact convex subset of a locally convex vector space. Suppose that either  $K$  is metrizable or that  $\partial K$  is closed. Then a  $*$ -representation  $\pi : C(K) \rightarrow \mathcal{B}(H)$  has the unique extension property relative to  $A(K)$  if and only if it is supported on  $\partial K$ .*

## 6. HYPERRIGIDITY OF FUNCTION SYSTEMS

In his seminal work on noncommutative dilation theory, Arveson [4] outlined many of the key ideas for a theory of dilations of operator algebras and operator systems. A central theme in his work is the notion of a boundary representation.



**Definition 6.1** (Arveson). Let  $A$  be a concrete operator system. An irreducible representation of  $C^*(A)$  is a *boundary representation* of  $A$  if it has the unique extension property relative to  $A$ . The *noncommutative Choquet boundary* is the set of all boundary representations of  $A$ .

Arveson conjectured that every concrete operator system has a noncommutative Choquet boundary with the property that the boundary representations provide a completely isometric representation of the operator system. Arveson's conjecture was established in the separable case by Arveson himself [6], and in the general case by the authors [11].

Motivated both by the fundamental role of the classical Choquet boundary in classical approximation theory, and by the importance of approximation in the contemporary theory of operator algebras, Arveson [7] introduced hyperrigidity as a form of approximation that captures many important operator-algebraic phenomena.

**Definition 6.2** (Arveson). A concrete operator system  $A$  that generates a  $C^*$ -algebra  $C$  is *hyperrigid* if whenever  $\pi : C \rightarrow \mathcal{B}(H)$  is a nondegenerate  $*$ -representation and  $\phi_n : C \rightarrow \mathcal{B}(H)$  is a sequence of unital completely positive maps with the property that

$$\lim_n \|\phi_n(a) - \pi(a)\| = 0 \quad \text{for all } a \in A,$$

then

$$\lim_n \|\phi_n(c) - \pi(c)\| = 0 \quad \text{for all } c \in C.$$

This definition is very useful once it is established, but there are equivalent formulations that are easier to verify.

**Theorem 6.3** (Arveson). *Let  $A$  be a concrete operator system that generates a  $C^*$ -algebra  $C$ . Then  $A$  is hyperrigid if and only if every  $*$ -representation  $\pi : C \rightarrow \mathcal{B}(H)$  has the unique extension property relative to  $A$ .*

It follows immediately from Definition 6.1 and Theorem 6.3 that a necessary condition for a concrete operator system to be hyperrigid is that every irreducible representation of the  $C^*$ -algebra it generates is a boundary representation. Arveson conjectured [7, Conjecture 4.3] that this is the only obstruction to hyperrigidity.

**Conjecture 6.4** (Arveson). *Let  $A$  be a concrete operator system that generates a  $C^*$ -algebra  $C$ . Then  $A$  is hyperrigid if and only if every irreducible  $*$ -representation of  $C$  is a boundary representation of  $A$ .*

A much more general problem than Conjecture 6.4 is to characterize the  $*$ -representations of a  $C^*$ -algebra with the unique extension property relative to an operator system that generates it. Using the results of the previous section, we will provide a complete solution to this more general problem for the case of a function system.

Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ . Theorem 2.2 implies that there is an order isomorphism  $\iota : A \rightarrow A(K)$ , where  $A(K)$  denotes the function system of continuous affine functions on the state space  $K = S(A)$  of  $A$ . Furthermore, there is a  $*$ -homomorphism  $q : C(K) \rightarrow C(X)$  such that  $\iota \circ q$  is the identity on  $A$ . Consider the adjoint  $q^* : C(X)^* \rightarrow C(K)^*$ . As in Section 2.3, identifying  $X$  and  $K$  with the corresponding point evaluations,  $q^*$  maps  $X$  into  $K$  and  $q^*(\partial_A X) = \partial K$ .

The irreducible representations of  $C(X)$  are precisely the point evaluations. For  $x \in X$ , the corresponding point evaluation has the unique extension property relative to  $A$  if and only if  $q^*(x)$  has the unique extension property relative to  $A(K)$ . By standard Choquet theory, this happens if and only if  $q^*(x) \in \partial K$ , and hence if and only if  $x$  belongs to the classical Choquet boundary  $\partial_A X$  of  $A$ . Thus Arveson's noncommutative Choquet boundary of  $A$  coincides with the classical Choquet boundary of  $A$  in the commutative setting. If  $A$  is hyperrigid, then we have already observed that every irreducible representation of  $C(X)$  is a boundary representation for  $A$ . Thus, in this case  $\partial_A X = X$ .

More generally, every  $*$ -representation of  $C(X)$  gives rise to a  $*$ -representation of  $C(K)$  via composition with  $q$ , and Theorem 5.5 completely characterizes representations of  $C(K)$  with the unique extension property relative to  $A(K)$ . While it is not immediately clear that this can be used to characterize  $*$ -representations of  $C(X)$  with the unique extension property relative to  $A$ , the next result shows that this is indeed the case.

From the operator algebraic viewpoint, the result reduces to the fact that the maximality of a unital completely positive map is an intrinsic property of the operator system, and does not depend on the  $C^*$ -algebra that it generates.

**Theorem 6.5.** *Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ . Let  $K = S(A)$  denote the state space of  $A$ , let  $\iota : A \rightarrow A(K)$  denote the canonical order isomorphism, and let  $q : C(K) \rightarrow C(X)$  denote the canonical quotient map as in Theorem 2.2. A  $*$ -representation  $\pi : C(X) \rightarrow \mathcal{B}(H)$  has the unique extension property relative to  $A$  if and only if the  $*$ -representation  $\pi \circ q : C(K) \rightarrow \mathcal{B}(H)$  has the unique extension property relative to  $A(K)$ .*

*Proof.* It is clear that if  $\pi \circ q$  has the unique extension property relative to  $A(K)$ , then  $\pi$  has the unique extension property relative to  $A$ . For the converse, suppose that  $\pi \circ q$  does not have the unique extension property relative to  $A(K)$ . Then by Proposition 5.3, the restriction  $\pi \circ q|_{A(K)}$  is not maximal. Thus there is unital completely positive map  $\phi : A(K) \rightarrow B(L)$  and an isometry  $V : H \rightarrow L$  such that  $VH$  is not invariant for  $\phi(A(K))$  and  $\pi \circ q(b) = V^*\phi(b)V$  for all  $a \in A(K)$ .

The order isomorphism  $\iota$  satisfies  $q \circ \iota(a) = a$  for all  $a \in A$ . Thus, defining  $\psi : A \rightarrow B(L)$  by  $\psi(a) = \phi \circ \iota(a)$  for  $a \in A$ , it follows from above that  $VH$  is not invariant for  $\psi(A)$  and  $\pi(a) = V^*\psi(a)V$  for all  $a \in A$ . In other words,  $\psi$  is a non-trivial dilation of the restriction  $\pi|_A$ . Therefore, by Proposition 5.3,  $\pi$  does not have the unique extension property relative to  $A$ .  $\square$

The next result follows immediately from Theorem 6.5 and Theorem 5.5.

**Corollary 6.6.** *Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ , and let  $\mu \in M^+(X)$  be a measure with corresponding GNS representation  $\pi_\mu : C(X) \rightarrow \mathcal{B}(L^2(\mu))$ . Let  $K = S(A)$  denote the state space of  $A$ , let  $\iota : A \rightarrow A(K)$  denote the canonical order isomorphism, and let  $q : C(K) \rightarrow C(X)$  denote the canonical quotient map as in Theorem 2.2. Then  $\pi_\mu$  has the unique extension property relative to  $A$  if and only if the pushforward measure  $\mu \circ (q^*)^{-1} \in M^+(K)$  is a boundary measure.*

For separable function systems and function systems with closed Choquet boundary, combining Theorem 6.5 with Corollary 5.6 yields an intrinsic characterization of  $*$ -representations with the unique extension property.

**Corollary 6.7.** *Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ . Suppose that either  $A$  is separable or that the Choquet boundary  $\partial_A X$  of  $A$  is closed. Then a  $*$ -representation of  $C(X)$  has the unique extension property relative to  $A$  if and only if it is supported on  $\partial_A X$ .*

For a measure  $\mu \in M^+(X)$  supported on the Choquet boundary  $\partial_A X$ , the corresponding GNS representation  $\pi_\mu : C(X) \rightarrow \mathcal{B}(L^2(\mu))$  can be viewed as a direct integral of boundary representations of  $A$ . Thus one interpretation of Corollary 6.7 is that, in many cases, a representation of  $C(X)$  has the unique extension property if and only if it is a direct integral of boundary representations of  $A$ .

If  $A$  is hyperrigid, then we have already observed that  $\partial_A X = X$ . Conversely, if  $\partial_A X = X$ , then in particular  $\partial_A X$  is closed. In this

case, every  $*$ -representation of  $C(X)$  is supported on  $X = \partial_A X$ . Thus Corollary 6.7 applies and we conclude that every  $*$ -representation of  $C(X)$  has the unique extension property relative to  $A$ , which yields a solution to Conjecture 6.4 in the commutative case.

**Corollary 6.8** (Hyperrigidity). *Let  $A$  be a concrete function system that generates a commutative  $C^*$ -algebra  $C(X)$ . Then  $A$  is hyperrigid if and only if  $X = \partial_A X$ .*

## 7. APPLICATIONS TO APPROXIMATION THEORY

A classical result from approximation theory is the following theorem of Korovkin.

**Theorem 7.1** (Korovkin). *If  $\phi_n : C[a, b] \rightarrow C[a, b]$  are positive maps such that*

$$\lim_{n \rightarrow \infty} \|\phi_n(g) - g\| = 0 \quad \text{for all } g \in \{1, x, x^2\},$$

*then*

$$\lim_{n \rightarrow \infty} \|\phi_n(f) - f\| = 0 \quad \text{for all } f \in C[a, b].$$

We will consider a very general extension of Korovkin's theorem due to Šaškin [20] for the setting of a unital commutative  $C^*$ -algebra. Šaškin's theorem establishes a connection between approximation-theoretic properties and the theory of the classical Choquet boundary (see Section 2.3). Before stating the theorem, we require the following definition.

**Definition 7.2.** Let  $C(X)$  be the  $C^*$ -algebra of continuous functions on a compact Hausdorff space  $X$ . A subset  $G \subset C(X)$  is a *Korovkin set* if whenever  $\phi_n : C(X) \rightarrow C(X)$  is a sequence of positive linear maps satisfying

$$\lim_{n \rightarrow \infty} \|\phi_n(g) - g\| = 0 \quad \text{for all } g \in G,$$

*then*

$$\lim_{n \rightarrow \infty} \|\phi_n(f) - f\| = 0 \quad \text{for all } f \in C(X).$$

**Theorem 7.3** (Šaškin). *Let  $C(X)$  be the  $C^*$ -algebra of continuous functions on a compact metric space  $X$ . A subset  $G \subset C(X)$  that separates points and contains 1 is a Korovkin set if and only if  $\partial_A X = X$ , where  $A = \overline{\text{span}(G \cup G^*)}$  denotes the function system generated by  $G$ .*

As an application of the results from the previous section, we obtain a significantly stronger version of Šaškin's theorem, where  $X$  is not required to be metrizable and where, more significantly, the maps are permitted to have noncommutative range.

**Definition 7.4.** Let  $C(X)$  be the  $C^*$ -algebra of continuous functions on a compact Hausdorff space  $X$ . A subset  $G \subset C(X)$  is a *strong Korovkin set* if whenever  $\pi : C(X) \rightarrow \mathcal{B}(H)$  is a  $*$ -representation and  $\phi_n : C(X) \rightarrow \mathcal{B}(H)$  is a sequence of positive linear maps satisfying

$$\lim_n \|\phi_n(g) - \pi(g)\| = 0 \quad \text{for all } g \in G,$$

then

$$\lim_n \|\phi_n(f) - \pi(f)\| = 0 \quad \text{for all } f \in C(X).$$

**Theorem 7.5.** Let  $C(X)$  be the  $C^*$ -algebra of continuous functions on a compact Hausdorff space  $X$ . Let  $G \subset C(X)$  be a subset that separates points and contains 1. Then the following are equivalent:

- (1)  $G$  is a strong Korovkin set.
- (2)  $G$  is a Korovkin set.
- (3)  $\partial_A X = X$ .
- (3') Every point  $x \in X$  has a unique representing measure for  $A$ .

Here  $A = \overline{\text{span}(G \cup G^*)}$  denotes the function system generated by  $G$ .

*Proof.* The implication (1)  $\Rightarrow$  (2) is trivial, and the implication (2)  $\Rightarrow$  (3) follows from Theorem 7.3. The implication (3)  $\Rightarrow$  (1) follows from Corollary 6.8, and the equivalence (3)  $\Leftrightarrow$  (3') is a standard fact from Choquet theory.  $\square$

**Corollary 7.6.** Let  $C(X)$  be the  $C^*$ -algebra of continuous functions on a compact Hausdorff space  $X$ . Let  $G \subset C(X)$  be a unital subset with the property that for every pair of distinct points  $x, y \in X$ , there is a non-negative function  $g \in A = \overline{\text{span}(G \cup G^*)}$  such that  $0 = g(x) < g(y)$ . Then  $G$  is a strong Korovkin set.

*Proof.* Let  $\mu$  be a probability measure with  $\mu \neq \delta_x$ . Then there is another point  $y \in X$  in the support of  $\mu$ . The function  $g \in A$  in the hypothesis is strictly positive on some neighbourhood of  $y$ . Hence  $\int g d\mu > 0 = g(0)$ . So  $\mu$  is not a representing measure for  $x$ . Hence every point in  $X$  has a unique representing measure. By Theorem 7.5,  $G$  is a strong Korovkin set.  $\square$

More generally, instead of considering sets that contain 1, we can consider sets that contain a strictly positive function.

**Corollary 7.7.** Let  $C(X)$  be the  $C^*$ -algebra of continuous functions on a compact Hausdorff space  $X$ . Let  $M \subset C(X)$  be a subset that separates points, let  $g_0$  be a strictly positive function, and set  $G = \{g_0\} \cup g_0 M$ . Then the following are equivalent:

- (1)  $G$  is a strong Korovkin set.

- (2)  $G$  is a Korovkin set.
- (3)  $\partial_A X = X$ .
- (3') Every point  $x \in X$  has a unique representing measure for the function system  $A = \overline{\text{span}\{1, M, M^*\}}$  generated by  $M$ .

*Proof.* Let  $G' = g_0^{-1}G = \{1\} \cup M$ . Then  $1 \in G'$  and  $G'$  separates points of  $X$ . The key observation is that there is a correspondence between sequences  $\phi_n : C(X) \rightarrow \mathcal{B}(H)$  of positive linear maps satisfying  $\lim_n \|\phi_n(g) - \pi(g)\| = 0$  for all  $g \in G$ , and sequences  $\phi'_n : C(X) \rightarrow \mathcal{B}(H)$  of positive linear maps satisfying  $\lim_n \|\phi'_n(g) - \pi(g)\| = 0$  for all  $g \in G'$ . The correspondence is given by

$$\phi'_n(f) = \pi(g_0^{-1/2})\phi_n(g_0 f)\pi(g_0^{-1/2}) \quad \text{for } f \in C(X),$$

and

$$\phi_n(f) = \pi(g_0^{1/2})\phi'_n(g_0^{-1}f)\pi(g_0^{1/2}) \quad \text{for } f \in C(X).$$

The result now follows from Theorem 7.5.  $\square$

**Example 7.8.**

- (1)  $G = \{1, x^a, x^b\}$  is a strong Korovkin set for  $C[a, b]$  if  $1 \leq a < b$ .
- (2)  $G = \{1, e^{ax}, e^{bx}\}$  is a strong Korovkin set for  $C[a, b]$  if  $0 < a < b$ .
- (3) If  $X$  is a compact convex subset of  $\mathbb{R}^n$  and  $\{f_1, \dots, f_k\}$  in  $C_{\mathbb{R}}(X)$  separate points, then  $G = \{1, f_1, \dots, f_k, \sum_{i=1}^k f_i^2\}$  is a strong Korovkin set. To see this, note that for each  $x \in X$ , the function  $g(y) = \sum_{i=1}^n (f_i(y) - f_i(x))^2$  belongs to  $A = \text{span}(G)$ , and  $0 = g(x) < g(y)$  for all  $y \neq x$ . Apply Corollary 7.6.

The next result is a variant of a recent result of Brown [9]. Brown's result deals with the weak and strong operator topologies, and his argument is both classical and non-trivial.

**Theorem 7.9.** *Let  $g$  be a strictly convex continuous function on a bounded interval  $I \subset \mathbb{R}$ . Let  $B_n \in \mathcal{B}(H)$  be a net of self-adjoint operators with  $\sigma(B_n) \subset I$  such that*

$$\lim_{n \rightarrow \infty} \|f(B_n) - f(B)\| = 0 \quad \text{for all } f \in \{1, x, g\}.$$

*Then*

$$\lim_{n \rightarrow \infty} \|f(B_n) - f(B)\| = 0 \quad \text{for all } f \in C(I).$$

*Proof.* Each operator  $B_n$  determines a  $*$ -representation  $\pi_n : C(I) \rightarrow \mathcal{B}(H)$  by

$$\pi_n(f) = f(B_n) \quad \text{for } f \in C(I).$$

Define a function system  $A = \text{span}\{1, x, g\} \subset C(I)$ , and let  $\phi_n = \pi_n|_A$ . Then  $\phi_n$  is a completely positive map.

The state space  $K$  of  $A$  can be identified with the convex hull of the graph

$$G(g) := \{(x, g(x)) : x \in [a, b]\}.$$

This is a closed set, and the strict convexity of  $g$  implies that  $\partial K = G(g)$ . Therefore, by Corollary 6.8,  $A$  is hyperrigid and the result now follows.  $\square$

The following corollary was established by Arveson in the finite dimensional case, and by Brown in general. It follows immediately from Theorem 7.9 by taking  $B_n = B$  for all  $n$ . Arveson showed that strict convexity of  $g$  (or  $-g$ ) is also a necessary condition.

**Corollary 7.10** (Brown). *Let  $g$  be a strictly convex continuous function on a bounded interval  $I \subset \mathbb{R}$ , and let  $B \in \mathcal{B}(H)$  be a self-adjoint operator with  $\sigma(B) \subseteq [a, b]$ . Suppose that  $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  is a completely positive map such that*

$$\phi(g(B)) = g(\phi(B)) \quad \text{for all } g \in \{1, x, g\}.$$

*Then*

$$\phi(f(B)) = f(\phi(B)) \quad \text{for all } f \in C[a, b].$$

**Example 7.11.** We now consider an example which shows that permitting the maps in Definition 7.4 to have noncommutative range adds considerable complexity.

Let  $S^5$  denote the unit sphere of  $\mathbb{R}^6$ , and consider the function system  $A(S^5)$  of affine functions. Let  $\{e_1, \dots, e_6\}$  denote the coordinate functions that form the standard basis for  $\mathbb{R}^6$ . Then  $A(S^5) = \text{span}\{1, e_1, \dots, e_6\}$ . Clearly every point of  $S^5$  is an extreme point. Thus  $A(S^5)$  is hyperrigid by Corollary 6.8.

The hyperrigidity of  $A(S^5)$  also follows from ideas in [7]. Suppose that  $\pi : C(S^6) \rightarrow \mathcal{B}(H)$  is a  $*$ -representation and that  $\phi : C(S^5) \rightarrow \mathcal{B}(H)$  is a completely positive extension of the restriction  $\pi|_{A(S^5)}$ . By the Kadison-Schwarz inequality for completely positive maps,  $\phi(e_i^2) \geq \phi(e_i)^2 = \pi(e_i)^2$  for each  $i$ . Thus  $1 = \sum_i \phi(e_i^2) \geq \sum_i \phi(e_i)^2 = 1$ , which implies that  $\phi(e_i^2) = \pi(e_i)^2$ . Therefore,  $e_i$  belongs to the multiplicative domain of  $\phi$ ,

$$\{f \in C(S^5) : \phi(|f|^2) = \phi(f)^* \phi(f) = \phi(f) \phi(f)^*\}.$$

This set is always a  $C^*$ -algebra by a result of Choi (see e.g. [18, Theorem 3.19]). Since  $A(S^5)$  generates  $C(S^5)$ , it follows that  $\phi = \pi$ . This verifies that  $\pi$  unique extension property relative to  $A(S^5)$ .



Although the second argument is elementary, it is non-classical in the sense that it utilizes a matrix inequality in the form of the Kadison-Schwarz inequality. Therefore, it is natural to ask whether such non-classical methods are actually required. The following heuristic argument suggests that they are.

Let  $m$  denote the unique rotation invariant probability measure on  $S^5$ , and let  $\pi_m : C(S^5) \rightarrow L^2(m)$  denote the corresponding GNS representation. Let  $\phi : C(S^5) \rightarrow \mathcal{B}(L^2(m))$  be a completely positive extension of the restriction  $\pi_m|_{A(S^5)}$ .

It is a standard result in Choquet theory that for every real-valued function  $f \in C(S^5)$ ,

$$f(x) = \inf\{a(x) : a \in A(S^5), f \leq a\} \quad \text{for all } x \in S^5.$$

Furthermore, for every affine function  $a \in A(S^5)$ , the Kadison-Schwarz inequality implies  $\pi_m(a^2) = \phi(a)^2 \leq \phi(a^2)$ . Therefore, one might hope to prove that  $\phi = \pi$  by showing that if  $T \in \mathcal{B}(H)$  satisfies

$$\pi(a^2) \leq T \leq \pi(b) \quad \text{for all } b \in \{b \in A(S^5) \mid a^2 \leq b\},$$

then  $T = \pi(a)^2$ . This would imply  $\phi(a^2) = \pi(a)^2$ , and arguing as above using the multiplicative domain would imply  $\phi = \pi$ .

However, the result cannot be established in this manner. To see this, take  $a = e_1$ . Then by some non-trivial computations that we do not reproduce here,

$$\pi(e_1^2) + \frac{1}{2}1_m 1_m^* \leq \pi(b) \quad \text{for all } b \in \{b \in A(S^5) : e_1^2 \leq b\},$$

where  $1_m 1_m^* \in \mathcal{B}(L^2(m))$  denotes the rank one operator defined by  $1_m 1_m^*(f) = \langle f, 1_m \rangle 1_m$ . Letting  $B = \text{span}\{A, e_1^2\}$ , it follows that the map  $\psi : B \rightarrow \mathcal{B}(L^2(m))$  given by

$$\psi(a + t e_1^2) = \pi(a + t e_1^2) + \frac{t}{2} 1_m 1_m^* \quad \text{for } a \in A, t \in \mathbb{C}$$

is a positive extension of  $\pi|_{A(S^5)}$  that differs from  $\pi|_B$ .

Necessarily, the map  $\psi$  is not completely positive. For, if it were, it would extend to a unital (completely) positive map on  $C(S^5)$  that differs from  $\pi$ , contradicting the fact that  $A(S^5)$  is hyperrigid. The conclusion is that matrix inequalities are critical to the analysis of this problem, even for commutative systems.

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